PHYSICAL REVIEW E VOLUME 59, NUMBER 1 JANUARY 1999

# Control of transients in ''lethargic'' systems

G. L. Lippi, P. A. Porta,* L. M. Hoffer, and H. Grassi
*Institut Non Linéaire de Nice, UMR 6618 CNRS, Université de Nice-Sophia Antipolis, 1361 Route des Lucioles, F-06560 Valbonne, France*

(Received 29 May 1998; revised manuscript received 7 October 1998)

We introduce a nonfeedback technique for the control of the transition between two steady states in a dynamical system with two very different time scales. We exploit the phase space properties by applying a series of discrete constant levels in a control parameter, and are able to control both the delay time and the height of the overshoot in the system's response. The results of the numerical integration of a ''paradigmatic'' model are in good qualitative agreement with experimental results obtained in a Class *B* laser.
[S1063-651X(99)50901-X]

PACS number(s): 05.45.−a, 42.60.Mi

A wealth of investigation in the past two decades has yielded a considerable understanding of the properties of dynamical systems. However, one aspect of their evolution has been largely ignored: the transition from one state into another, and the possibility of controlling these transients. Only very recently has a nonlinear control technique for tayloring the evolution between two states of a dynamical system been proposed [1].

We consider a ''paradigmatic'' dynamical system described by two variables: one slow and one fast. The most interesting transition, but also the most difficult to control, is the one starting from an initial fixed point where the fast variable is nearly zero. In these systems, after the bifurcation point is crossed, the representative point in phase space moves ''slowly'' along the unstable manifold and at a later time moves away very quickly. This ''fast'' rate depends on the time constant of the fast variable and the distance from the bifurcation point. A standard feedback technique applied to the fast variable would not be able to control such a transition, because initially its value is too small to be measured, and later on it changes too rapidly. Feedback applied to the slow variable is possible, but would not be able to cope with the fast evolution. Instead, we base our control technique on the features of the phase space and modulate the control parameter according to a predetermined pattern. We therefore act directly on the slow variable, producing a customized trajectory in phase space.

This control technique is relatively easy to implement experimentally, as we successfully demonstrate on a Class-*B* laser [2]. It makes a *strong reduction* in the *transition time* possible in ''lethargic'' systems, i.e., dynamical systems governed by variables with very different time constants. We also show that this method can *reduce* the *amplitude* of the transient overshoot.

Let us consider a dynamical system modeled by two variables $X$ and $Y$, with time scales $(2\epsilon)^{-1}$ and $\epsilon$, respectively, where $\epsilon$ is a small quantity. The dynamics of this system are determined by

$$\frac{dX}{d\tau}=\frac{1}{2\epsilon}(XY-X), \tag{1a}$$

$$\frac{dY}{d\tau}=\epsilon[1-Y-Y|X|^2+\lambda(\tau)], \tag{1b}$$

where $X$ represents a complex variable, $Y$ a real variable, $\tau$ is time (normalized), and $\lambda(\tau)$ is the time-dependent control parameter. This is a generic model that captures the main features of many dynamical systems: Class *B* lasers [2], population dynamics [3], and ignition reactions [4]. As such, it does not account for the details of the behavior of any of these systems, but accurately reproduces their general features.

Equations (1) have two stationary solutions ($X_l=0$, $Y_l=1+\lambda$) and ($|X_h|^2=\lambda$, $Y_h=1$), only one of which is stable for each value of the control parameter. The exchange of stability occurs for $\lambda=0$, the bifurcation point. A typical time evolution in response to a sudden variation from a negative to a positive value of $\lambda$ is shown in Fig. 1 (long-dashed line).

In what follows, we will discuss the control of the deterministic system only. In most Class *B* lasers, indeed, noise is quite small and its influence is only minor. Furthermore, in-

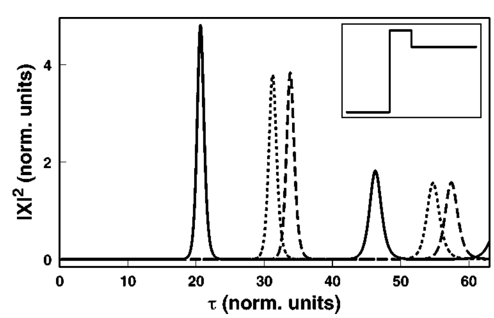


FIG. 1. Time evolution of Eqs. (1) for a ''controlled'' transition. Inset: generic control pattern for $N=2$. $\lambda_0=0.9$, $\lambda_2=1.2$, and $\tau_1=20$. The dotted line shows a reduction in delay with correspondingly smaller peak overshoot ($\lambda_1=1.25$). The solid line shows a strong reduction in delay time and an increase in the peak amplitude ($\lambda_1=1.5$). The dashed line is the uncontrolled transition obtained with a simple square pulse.

*Present address: Physics Department, National University of Ireland, University College, Cork, Ireland.

 

spection of the model's properties suggest that noise on the initial conditions is *linearly* transferred onto the overshoot amplitude and onto the delay (cf. also [5]), causing only a small stochastic spread in the features we are controlling.

The generic control scheme consists of a succession of $N$ discrete steps in the control parameter. This corresponds to a sequence of fixed points as interim goals, leading the system along a customized trajectory which differs from the freely evolving one. We choose

$$\lambda(\tau)=\lambda_0\theta(-\tau_0)+\sum_{j=1}^{N}(\lambda_j-\lambda_{j-1})\theta(\tau_{j-1}), \qquad (2)$$

where $\lambda_0$ is the initial value, $\tau_0$ is the time at which the initial state is abandoned [6], the control parameter levels $\lambda_{1\ldots N-1}$ and the durations $(\tau_j-\tau_{j-1})$ can be chosen arbitrarily, $\lambda_N$ is the final value of the control parameter (we impose $\tau_N\to\infty$), and $\theta(\tau)$ represents the Heaviside function at time $\tau$.

For an approximate analytical treatment, we divide the time evolution up into distinct regions [7]. The first region is made up of those control steps during which $X(\tau)$ is negligibly small, and ends at $\tau^*$, when $X(\tau)$ reaches an arbitrary threshold value, $X_{th}$. In the second region, the full set of equations must be integrated. During the overshoot, however, $X$ is very large and we can, in this third region, perform an approximate analytical calculation which yields the peak height, $|X_m|^2$.

A general solution of Eqs. (1) for $X$ always negligibly small can be obtained for each time interval $\tau_{j-1}<\tau<\tau_j$ in the form

$$Y(\tau)=(Y_{j-1}-Y_{\infty,j})e^{-\epsilon(\tau-\tau_j)}+Y_{\infty,j}, \qquad (3)$$

where

$$Y_j=Y(\tau_j)=(Y_{j-1}-Y_{\infty,j})e^{-\epsilon(\tau_j-\tau_{j-1})}+Y_{\infty,j}, \qquad (4)$$

and $Y_{\infty,j}=1+\lambda_j$ is the asymptotic value.

We choose the $\lambda_j$'s and $\tau_j$'s such that $X_{th}$ is reached during the $N$th step. Using Eq. (3), we obtain

$$Y(\tau^*)=(Y_{N-1}-Y_{\infty,N})e^{-\epsilon(\tau^*-\tau_{N-1})}+Y_{\infty,N}. \qquad (5)$$

From this point onward, the dynamical evolution of the system is governed by the full Eqs. (1). A numerical simulation shows the characteristic sudden growth of the $X$ variable after a certain delay time, followed by an overshoot with oscillatory relaxation towards the final fixed point (Fig. 1, long-dashed line), i.e., a spiraling in phase space.

Following the technique outlined in [7], when $X$ is very large, we can solve Eq. (1b) and write the maximum for $X$, at time $\tau_m$, as

$$|X_m|^2=|X_{th}|^2+\frac{1}{\epsilon^2}\left\{Y(\tau^*)-1+\ln\left(\frac{1}{Y(\tau^*)}\right)\right\}, \qquad (6)$$

where we have substituted $Y(\tau_m)=1$ and, as in [7], have kept the lowest order [8]. Since $Y(\tau^*)$ depends on the $\lambda_j$'s and $\tau_j$'s, we can change $|X_m|^2$ by modifying the heights and durations of the control steps.

We remark that $X_{th}$ is arbitrary and uniquely defines $\tau^*$. Since $X$ is negligibly small until time $\tau^*$, we can neglect the bilinear term in Eq. (1b) and obtain, by formal integration of Eq. (1a),

$$|X_{th}|^2=|X(\tau_0)|^2\exp\left\{\frac{1}{\epsilon}\int_{\tau_0}^{\tau^*}[Y(\tau')-1]d\tau'\right\}, \qquad (7)$$

which can be solved for $\tau^*$. Note that since $Y(\tau')$ depends explicitly on the $\lambda_j$'s and $\tau_j$'s, modifying these values can cause $X_{th}$ to occur for different values of $\tau^*$, thus controlling the delay time.

We now consider specifically $N=2$ and $\lambda_1>\lambda_2$ (inset of Fig. 1). Adding a higher control step forces $Y(\tau)$ to grow at a faster rate for a time $\tau_1$ [Eq. (3)], and therefore $X_{th}$ is reached at an earlier time $\tau^*$ [Eq. (7)]. This anticipation of $\tau^*$ is responsible for a decrease in the delay time for increasing $\lambda_1$ or $\tau_1$, because the interval $\tau_m-\tau^*$ only changes a small amount in comparison to the change in $\tau^*-\tau_0$. The amplitude of the overshoot, Eq. (6), instead, first decreases and then increases again. To show this more easily, we consider short values of $\tau_1$ ($\tau_1\ll\epsilon^{-1}$) such that the expansion of Eq. (4) to first order in $\epsilon$, substituted in Eq. (3), gives

$$Y(\tau^*)=Y_1[1-\epsilon(\tau^*-\tau_1)]+(1+\lambda_2)\epsilon(\tau^*-\tau_1). \qquad (8)$$

Substituting these expressions into Eq. (6), and searching for the minimum of $|X_m|^2$ with respect to $Y_1$, we obtain

$$Y_1=1-\lambda_2\epsilon(\tau^*-\tau_1), \qquad (9)$$

which determines the value of $Y(\tau_1)$ for the occurrence of the smallest possible overshoot (for fixed parameters). Figure 1 shows the time evolution of $|X|^2$. The dashed and dotted lines indicate the transitions without control and with nearly optimal control (for the minimum peak), respectively. Increasing $\lambda_1$ (or $\tau_1$) further reduces the delay at the expense of an increase in the peak height. This situation is shown by the solid line in Fig. 1.

We have performed an experimental verification of the technique, with two and then three levels of pump, on a low pressure, flowing gas-mixture, cw electrically pumped $CO_2$ laser [9]. A detailed presentation of the experimental features can be found in [10], here we limit ourselves to a brief discussion of the main points. The laser is run in single longitudinal and transverse mode, is kept tuned to resonance for all measurements (cf. [9] for details) and is allowed to relax to the same initial condition between successive repetitions of the turn-on. The main advantage of this experimental system is the possibility of applying a very fast commutation (much faster than the internal time constants) to the current that flows through the gas and pumps the laser. This is achieved with the help of a fast programmable function generator (LeCroy 9100, 5 ns minimum time resolution) and of a fast summing circuit that drives the low voltage side of the electronics controlling the laser current. In this way, it is possible to apply a sequence of pumping steps (two or three for these measurements) to the laser, in close analogy to the theoretical discussion. Figure 2 shows the evolution of the laser intensity for an uncontrolled transition (dashed line), and for a controlled one (solid line). Since about 100 $\mu$s are necessary for the pumping process to take effect in $CO_2$ la-

RAPID COMMUNICATIONS



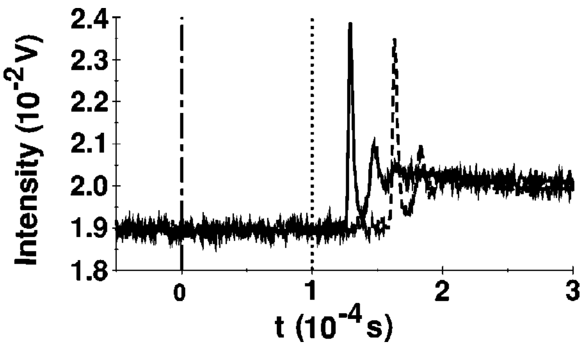


FIG. 2. Controlled turn-on in a $CO_2$ laser (experiment). $\lambda_0 \approx 0.9$, $\lambda_2 \approx 1.2$, and $t_1 = 20\ \mu s \leftrightarrow \tau_1 = 20$. The solid line ($\lambda_1 \approx 1.5$) shows a strong reduction in delay ($\approx 129\ \mu s$), compared to $\approx 163\ \mu s$ in the absence of control (dashed line). The dotted-dashed line marks time $\tau_0$. Only the delay to the right of the dotted line is of a dynamical nature (cf. text).

sers [11], we stress the considerable amount of reduction in *dynamical* delay ($\gtrsim 50\%$) that we achieve. We also remark that the rise time ($\approx 1\ \mu s$) of the laser intensity ($|X|^2$) would render a feedback control technique very difficult to implement. The qualitative agreement between our simple paradigm and the experiment confirms the effectiveness of the control method, and the strong reduction in the delay time achieved can have important applications.

We have not, however, shown a curve for a two-step controlled transition where the peak amplitude is reduced, because the reduction is too small (a few percent) to be unequivocally identified in a single event without noise reduction. When ensemble averages are taken, a reduction can be clearly seen, even in this case [12].

The reason for the overshoot in $X$ comes from the fact that growth away from the $Y$ axis in the $X$ direction, begins only *after* $Y$ has crossed 1, at a rate $dX/d\tau \sim (Y-1)$. However, if $Y>1$ when $X$ is no longer negligible, $dX/d\tau$ is large, and the system spirals into the final fixed point via wide excursions. This point is well illustrated by the phase space plot [Fig. 3(a), dashed line]. In order to produce a much tighter spiral and reduce the overshoot, we need to approach $X \approx X_h$ with $dX/d\tau$ small (i.e., $Y$ only slightly larger than 1).

We achieve this goal by adding a level to our control scheme (now $N=3$) with $0<\lambda_2<\lambda_3<\lambda_1$ [cf. inset in Fig. 3(a)]. During the interval $\tau_0<\tau<\tau_1$, $Y$ increases towards $Y_{\infty,1}$ and passes $Y_h=1$ at $\tau=\tau'$. During $\tau'<\tau<\tau_1$, $X$ grows rapidly away from $X=0$ [$dX/d\tau \sim Y(\tau)-1$]. At $\tau=\tau_1$, we change the control parameter to a new value: $\lambda_2=\eta>0$, $\eta \ll 1$. $Y(\tau_1)>Y_{\infty,2}$ and therefore $Y$ decays towards its steady state value for this interval: $(1+\lambda_2)$. At the same time, $dX/d\tau$ is positive, but decreasing. By adjusting $\lambda_2$ and $\tau_2$, we can bring $X$ close to $X_h$ when $Y$ is close to 1 [Fig. 3(b)]. The application of the last step, $\lambda_3$ at $\tau=\tau_2$, signals the beginning of the spiral, which cannot be completely suppressed (since the fixed point is a focus), but whose amplitude is now considerably reduced [Fig. 3(a)]. This can also be seen from Eq. (6): the closer $Y(\tau^*)$ is to 1, the smaller the value of $|X_m|^2$.

The reduction in the height of the overshoot comes at the expense of a lengthening of $\tau_2$, and therefore of the overall time delay. In order to achieve a value of $Y(\tau_2)=1+a\eta$, with $a \geqslant 1$—i.e., an arbitrary but small value above $Y_{\infty,2}$—, the duration of the second control step, $\tau_2$, to lowest approximation, must be

$$\tau_2=\tau_1+\frac{1}{\epsilon}\log\frac{Y_1-1-\eta}{(a-1)\eta}. \tag{10}$$

This shows that for $\eta \to 0$ (ideal transition) this time diverges. This time estimate for $\tau_2$ [Eq. (10)] does not take into account the dynamics of $|X|^2$, and is therefore only valid for moderately small values of $Y_1$ and for $\eta$ not too small.

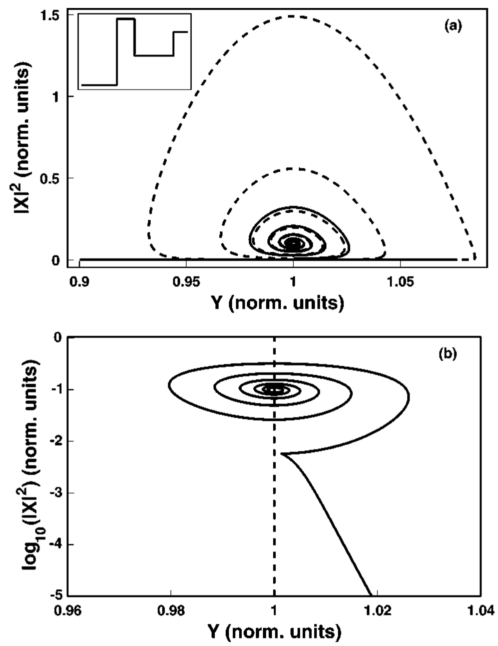


FIG. 3. Phase space plots of Eqs. (1) for $N=3$ (solid line), and for the uncontrolled transition $N=1$ (dashed line). Inset: generic control pattern for $N=3$. (a) full picture in linear scale, (b) detail in logarithmic scale. Parameters: $\lambda_1=1.25$, $\tau_1=14$, $\lambda_2=1.001$, $\tau_2=80$, and $\lambda_3=1.1$ (solid line); $\lambda_1=1.1$ (dashed line). $\lambda_0=0.9$ for both.

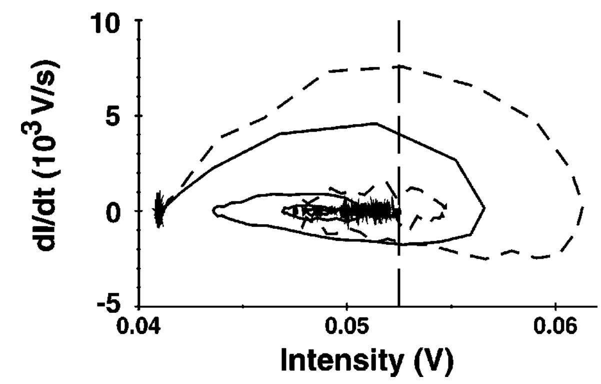


FIG. 4. Phase space plot of the $N=3$ transition in the laser (experiment): The dashed line represents the $N=1$ (uncontrolled) case; the solid line, the $N=3$ case. The vertical long-dashed line marks the final steady state towards which both transients evolve. The ''noisy-looking'' area at the core of the spiral is due to a gradual increase in intensity superimposed on the usual ''ringing'' behavior with which the laser relaxes to the steady state. Parameters: $\lambda_0 \approx 0.88$, $\lambda_1 \approx 1.15$, $\lambda_2 \approx 1.02$, $\lambda_3 \approx 1.12$, $t_1=40\ \mu s \leftrightarrow \tau_1=40$, and $t_2=160\ \mu s \leftrightarrow \tau_2=160$.

Figure 4 shows an experimental result with the control pattern of Fig. 3(a) (cf. inset). A comparison between the transition without control (dashed line) and with control (solid line) shows that the spiral amplitude is reduced by a factor of about 2 (at the price of an increase in the delay, $\approx 220\,\mu$s in total). Note that in both of the experimental examples, the laser is run far from the bifurcation point, so that noise does not play a strong role. This is not a very restrictive limitation, since noise is important only in a very narrow band around the lasing threshold [9]. The presence of noise implies, however, that in the $(N=3)$ control pattern we must keep $\lambda_2$ far enough from the bifurcation to keep the laser from being accidentally driven back to the $(X_l, Y_l)$ fixed point by a fluctuation. Hence, $\lambda_2 \gtrsim 5\times 10^{-3}$ [9].

Before concluding, we would like to mention some possible applications of this technique to lasers. The double control step $(N=2)$ could be very useful in reducing the turn-on time of semiconductor lasers (also Class $B$ systems [2]) for the optimization of data transmission. Given the very high speed requirements, our simple technique offers one of the best chances of a successful implementation [14]. However, its realization requires some further work, since it necessitates an extremely short rise time for the pump pulse (of the order of a fraction of a nanosecond) and at present there are no commercially available programmable function generators capable of that. On the positive side, the tolerances on the current stability are not very stringent for this technique and are not cause of particular concern. The triple $(N=3)$ control pattern, on the other hand, could be very useful in reducing the peak overshoot, and its potentially destructive effects, in high power lasers (solid-state, $CO_2$) or fiber amplifiers (all Class $B$ systems [2]).

In conclusion, we have shown that a simple, nonfeedback technique can control the transition between two steady states in a lethargic system. Experimental results confirm the effectiveness of the method.

We are grateful to A. Politi and J. R. Tredicce for discussions and encouragement.

[1] V. Petrov and K. Showalter, Phys. Rev. Lett. **76**, 3312 (1996).

[2] J. R. Tredicce, F. T. Arecchi, G. L. Lippi, and G. P. Puccioni, J. Opt. Soc. Am. B **2**, 173 (1985).

[3] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators* (Dover, New York, 1966).

[4] V. Barelko *et al.*, Dokl. Akad. Nauk SSSR **287**, 1373 (1986) [Sov. Phys. Dokl. **31**, 336 (1986)].

[5] S. Balle, M. San Miguel, N. B. Abraham, J. R. Tredicce, R. Alvarez, E. J. D'Angelo, A. Gambhit, K. S. Thanburg, and R. Roy, Phys. Rev. Lett. **72**, 3510 (1994).

[6] In the numerical integrations, we simulate the effect of noise, which kicks the system away from the initial fixed point, by adding a very small offset (typically $10^{-10}$ of the final steady state value) to the $X_l=0$ steady state.

[7] S. Balle, P. Colet, and M. San Miguel, Phys. Rev. A **43**, 498 (1991).

[8] This expression represents a good approximation for $|X_m|^2$ as long as $|X^*|^2 \ll |X_m|^2$ (true for Class $B$ lasers), otherwise a term of the form $\ln(|X^*|^2/|X_m|^2)$ must be added to Eq. (6).

[9] A simple model such as Eqs. (1) captures only the basic features of such a system. No quantitative comparison can therefore be expected [13], and in the experiment we must operationally find the optimal parameters.

[10] R. Meucci, M. Ciofini, and Peng-ye Wang, Opt. Commun. **91**, 444 (1992).

[11] H. Grassi, C. Green, L. M. Hoffer, G. L. Lippi, and J. R. Tredicce, Phys. Rev. A **50**, 805 (1994).

[12] W. J. Witteman, *The $CO_2$ Laser* (Springer, Berlin, 1987).

[13] P. A. Porta, L. M. Hoffer, H. Grassi, and G. L. Lippi, Int. J. Bifurcations Chaos Appl. Sci. Eng. (to be published).

[14] Similar ideas have been considered in the past for improving the performances of data transmission systems. Compare, e.g., L. Bickers, and L. D. Westbrook, IEE Proc. **133**, 155 (1986).